Struktura nieliniowych układów sterowania: linearyzacja statyczna i dynamiczna oraz formy normalne

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Komitet Automatyki i Robotyki PAN, 25 października, 2024

referat oparty o wspòlne badania naukowe z: J.P. Barbot, K. Kozłowski, Shunjie Li, C. Moog, F. Nicolau, M. Nowicki, J. Piasek, J. Ratajczak, K. Tchoń

Summary

What are normal forms?

- What are normal forms? It is not a precise mathematical notion.

Normal form $\equiv$ Simple form

- As few as possible parameters: real ones (if the system is linear), nonlinear ones (if the system is nonlinear)
- In general: we study a class of objects (systems) **Class** ;
- Elements of **Class** are denoted $\mathcal{O}$;
- We are given an equivalence relation $\mathcal{O} \sim \mathcal{O}'$;
- Often the equivalence relation is defined by a group of transformations $\mathcal{T}$: **Class** $\to$ **Class**, and $\mathcal{O} \sim \mathcal{O}'$ if there exists $T \in \mathcal{T}$ such that $\mathcal{O}' = T(\mathcal{O})$.
- Constructing normal forms is the problem: for any $\mathcal{O} \in$ **Class** (for some $\mathcal{O}$) find $NF \in$ **Class** such that

$$\mathcal{O} \sim NF \quad \text{and} \quad NF \quad \text{is simple.}$$

Example: classification of linear control systems

- **Class** $= \{$*Linear controllable systems* $\Lambda\}$, where $\Lambda: \ \dot{x} = Ax + Bu$
- $\mathcal{T} = \{T = (S, F, G)\}$, matrices of appropriate sizes
- $T(\Lambda) = \tilde{\Lambda}: \ \dot{z} = \tilde{A}z + \tilde{B}v$, where $z = Sx$, $u = Fx + Gv$, implying $\tilde{A} = S(A + BF)S^{-1}$, $\tilde{B} = SBG$.
- NF for any controllable linear system Λ is the Brunovský canonical form (Br), which reads in the single-input case as

$$\dot{z}_i = z_{i+1}, 1 \leq i \leq n-1$$
$$\dot{z}_n = v$$

- For $m = 1$, all systems Λ are equivalent to each other and equivalent to (Br). Why is it possible?
- (A, B) is given by $n^2 + n$ reals and $T = (S, F, G)$ is given by $n^2 + n + 1$ reals so $\mathcal{T}$ is reacher than **Class** and 1 real parameter is free. It defines the symmetry action $z_i \mapsto kz_i$, $v \mapsto kv$, $k \in \mathbb{R}^*$.

Class of control systems

Throughout our talk **Class** are nonlinear control systems

- finite-dimensional
- smooth
- time-continous
- control-affine

of the form

$$\Sigma \ : \ \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \quad x \in X \subset \mathbb{R}^n, u \in \mathbb{R}^m,$$

where

- $x \in X$, state space, an open subset of $\mathbb{R}^n$
- $u \in U$, set of control values, a subset of $\mathbb{R}^m$
- f and $g_1, \ldots, g_m$ are smooth vector fields on X
- state-dependent nonlinearities
- common in applications

Summary

The systems

$$\Sigma \ : \ \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) = f(x) + g(x)u, \quad x \in X \subset \mathbb{R}^n, u \in \mathbb{R}^m, \text{ and}$$

$$\tilde{\Sigma} \ : \ \dot{z} = \tilde{f}(z) + \sum_{i=1}^{m} v_i \tilde{g}_i(z) = \tilde{f}(z) + \tilde{g}(z)v, \quad z \in Z \subset \mathbb{R}^n, \ v \in \mathbb{R}^m \ \text{(not u)}$$

are feedback equivalent, shortly F-equivalent, if there exist

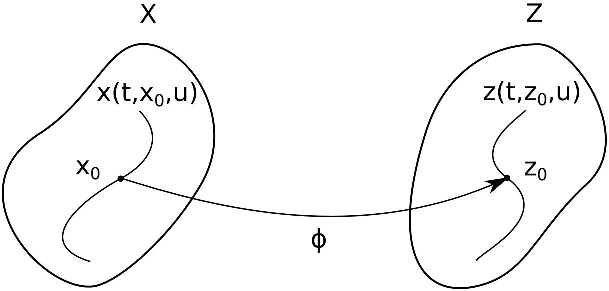
- a diffeomorphism $z = \Phi(x)$ and
- a control transformation $u = \alpha(x) + \beta(x)v$, where the matrix β is invertible (the control transformation between v and u is invertible),

such that

$$\frac{\partial \Phi}{\partial x} \cdot (f + \sum_{i=1}^{m} g_i \alpha_i)(x) = \tilde{f}(\Phi(x)), \quad \frac{\partial \Phi}{\partial x} \cdot (\sum_{i=1}^{m} \beta_j^i g_i)(x) = \tilde{g}_j(\Phi(x)).$$

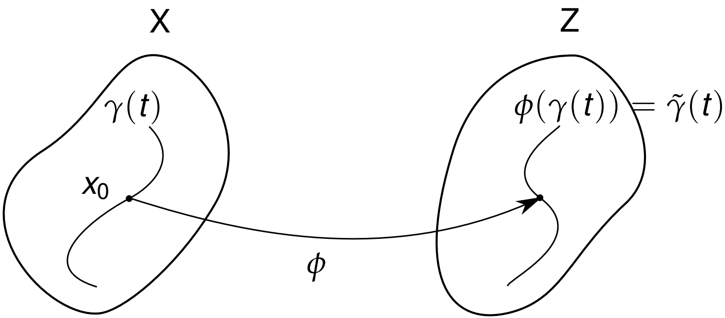
Why is F-equivalence interesting?

- A diffeomorphism is a map Φ such that
  - Φ is bijective
  - Φ and Φ^{-1} are C^k (C^∞)
- A (local) diffeomorphism defines a (local) nonlinear change of coordinates $z = \Phi(x)$
- A diffeomorphism Φ preserves trajectories.



The image under Φ of a trajectory of Σ is a trajectory of $\tilde{\Sigma}$ corresponding to the same control.

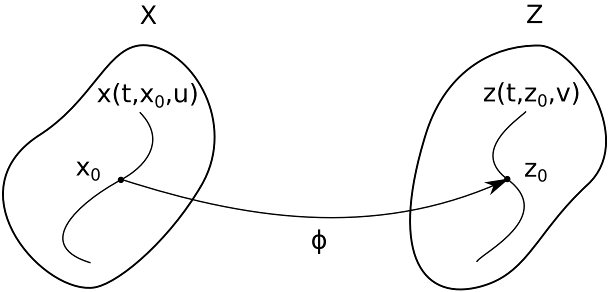
Does F-equivalence preserve trajectories?



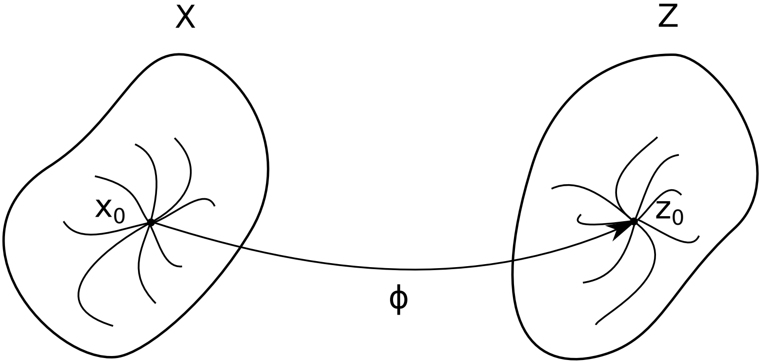
Is the image of a trajectory, via the diffeomorphism $z = \Phi(x)$, a trajectory?

Yes, the image of a trajectory of Σ, for a control $u(t)$, is a trajectory of $\tilde{\Sigma}$ corresponding to

$$u(t) = \alpha(x(t)) + \beta(x(t))v(t).$$

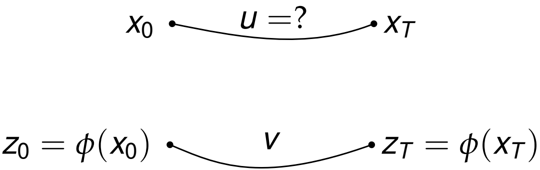


Therefore, F-equivalence preserves the set of all trajectories (the totality of trajectories)

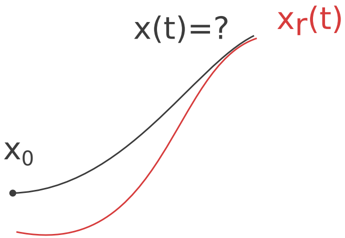


F-equivalence is thus interesting for all problems that depend on the set of all trajectories (and not on a particular parametrization with respect to control). Examples of such problems are: point-to-point controllability, trajectory tracking, stabilization.

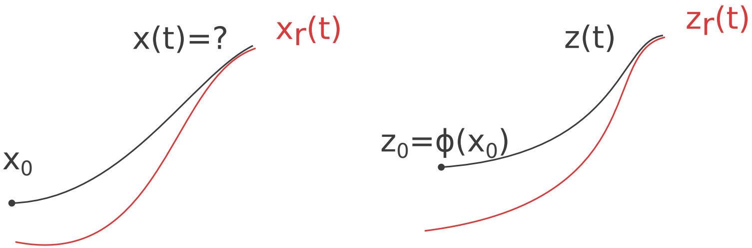
Point-to-point controllability



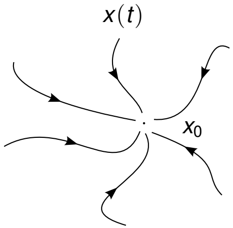
Trajectory tracking



Trajectory tracking



Stabilization



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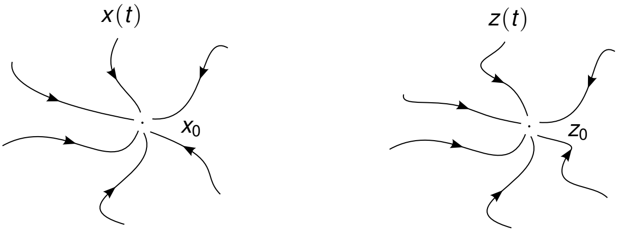

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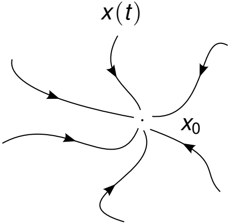
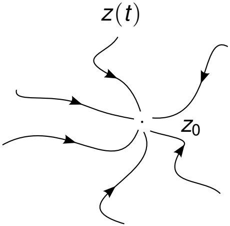
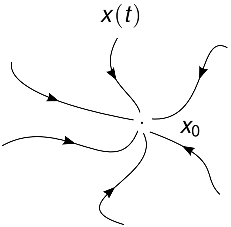


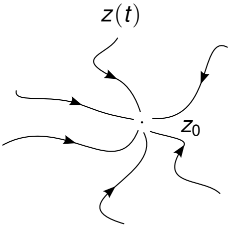


$v = \sigma(z)$ stabilizes (asymptotically)

$\dot{z} = (f + g\alpha) + g\beta v$

Stabilization





$u = \alpha + \beta\sigma$ stabilizes (asymptotically) $\dot{x} = f + gu$ $\Leftarrow$ $v = \sigma(z)$ stabilizes (asymptotically) $\dot{z} = (f + g\alpha) + g\beta v$

Problem When is Σ F-equivalent to a linear system, i.e., when do there exist $z = \Phi(x)$ and (α, β) transforming Σ into a linear system of the form

$$\dot{z} = Az + \sum_{i=1}^{m} v_i b_i, \quad z \in \mathbb{R}^n?$$

Normal forms of F-linearizable systems are thus simply normal forms of linear forms (Brunovský canonical form, controller form, controllability form etc.)

Why is F-linearization interesting?

- If we want to solve a control problem for Σ and Σ is F-equivalent to a linear system Λ, then
- transform Σ into Λ
- solve the problem for the linear system Λ
- transform the solution (via the inverse Φ^{-1} of Φ and the inverse control transformation $u = \alpha + \beta v$)
- we identify intrinsic nonlinearities

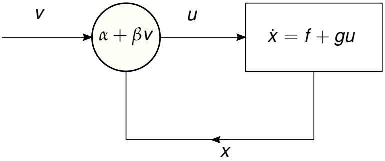
For control affine systems

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \quad x \in X$$

we apply $z = \Phi(x)$ and control-affine feedback transformation

$$u = \alpha(x) + \beta(x)v,$$

where the matrix β is invertible.



A little bit of geometry: Lie bracket

- Given two vector fields f and g on X, we define their Lie bracket as

$$[f, g](x) = \frac{\partial g}{\partial x}(x)f(x) - \frac{\partial f}{\partial x}(x)g(x)$$

  It is a new vector field on X.
- It is a geometric (invariant) object

$$\Phi_*[f, g] = [\Phi_* f, \Phi_* g], \text{ where } \Phi_* = \frac{\partial \Phi}{\partial x}$$

- It measures to what extent the flows of f and g do not commute

Consider the control system

$$\dot{x} = u_1 f(x) + u_2 g(x),$$

and apply the control strategy

$$\begin{array}{rcl} u_1 & = & 1 \\ u_2 & = & 0 \end{array} \quad s \in [0, t]$$

$$\begin{array}{rcl} u_1 & = & 0 \\ u_2 & = & 1 \end{array} \quad s \in [t, 2t]$$

$$\begin{array}{rcl} u_1 & = & -1 \\ u_2 & = & 0 \end{array} \quad s \in [2t, 3t]$$

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Denote by $\gamma_t^f(x_0) = x(t, x_0)$ the solution of the differential equation $\dot{x} = f(x)$ and by $\gamma_t^g(x_0) = x(t, x_0)$ the solution of the differential equation $\dot{x} = g(x)$, passing through x_0.

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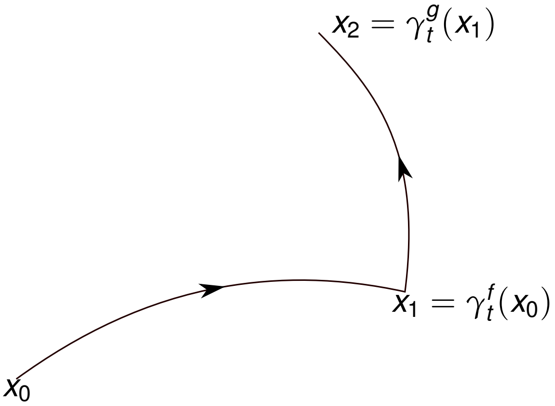

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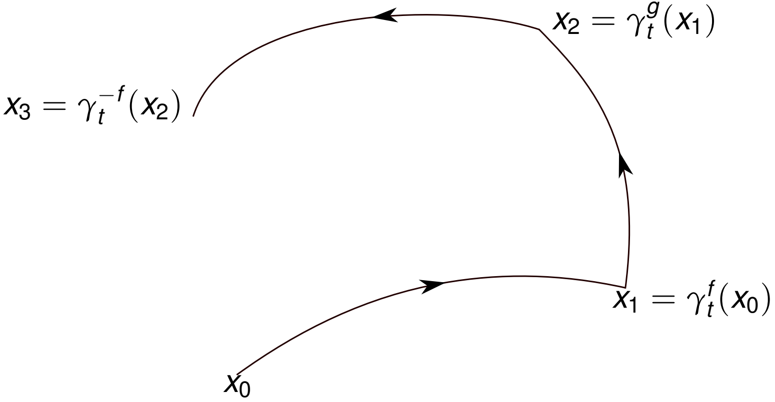

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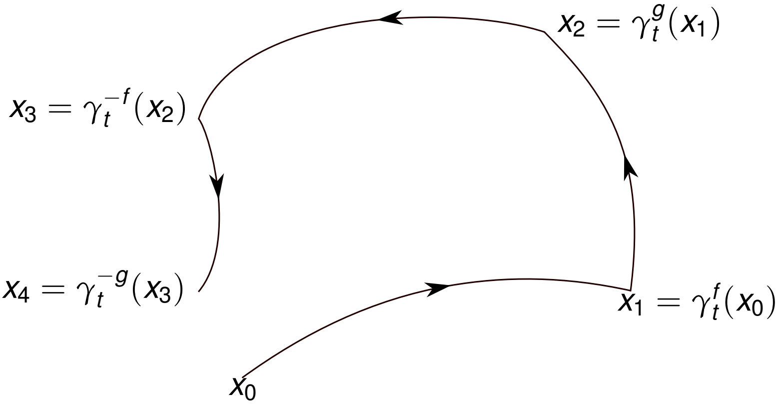

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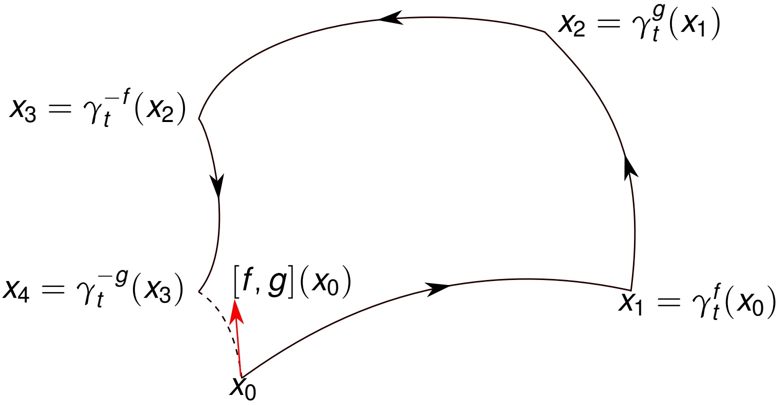


Notation

Define

$$\begin{aligned} ad_f^0 g &= g \\ ad_f g &= [f, g] \\ \text{and, inductively, } ad_f^k g &= [f, ad_f^{k-1} g] = [f, \ldots, [f, g]] \end{aligned}$$

For the single-input system

$$\dot{x} = f(x) + ug(x)$$

the Lie bracket $ad_f g = [f, g] = [f, f + g]$ measures to what extent the trajectories of f (corresponding to $u \equiv 0$) do not commute with those of $f + g$ (corresponding to $u \equiv 1$). In particular, for the linear system

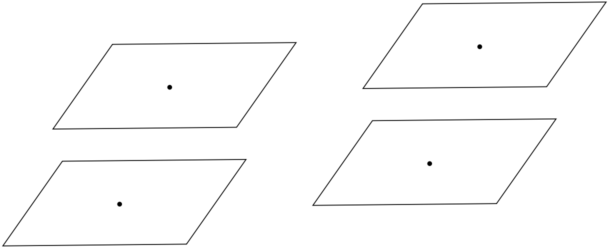
$$\dot{x} = Ax + bu$$

the Lie bracket $ad_{Ax} b = [f, g] = [Ax, Ax + b] = -Ab$ measures to what extent the trajectories of Ax (corresponding to $u \equiv 0$) do not commute with those of $Ax + b$ (corresponding to $u \equiv 1$).

More geometry: a distribution is a map assigning to any $x \in X$

$$x \longmapsto \mathcal{D}(x),$$

a linear subspace of the tangent space (the space of all tangent vectors at x or, equivalently, the space of all velocities at x)



- Let $\mathcal{D} = \operatorname{span}\{f_1, \ldots, f_k\}$ be a distribution spanned by vector fields
- $\mathcal{D}$ is involutive if $[f_i, f_j] \in \mathcal{D}$, for any $1 \le i, j \le k$
- Put $\mathcal{D}^j = \operatorname{span}\{ad_f^q g_i;\ 1 \le i \le m,\ 0 \le q \le j-1\}$

Theorem (Jakubczyk-R., Hunt-Su)

Σ is, locally around x_0, F-equivalent to a controllable linear system Λ if and only if

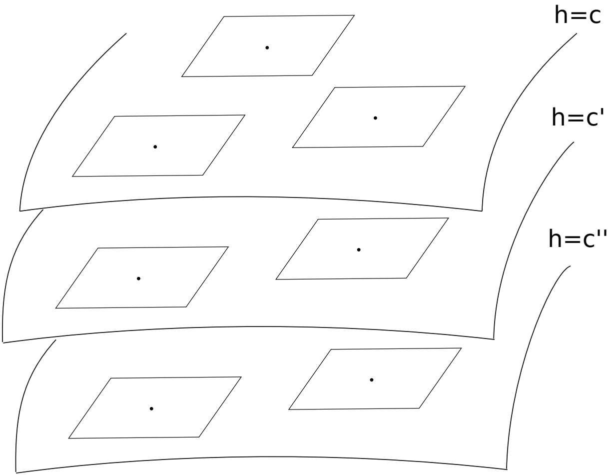
(FL1) $\dim \mathcal{D}^j(x)$*=const.*

(FL2) $\dim \mathcal{D}^n(x) = n$

(FL3) $\mathcal{D}^j$ *are involutive, for* $0 \le j \le n$

- (FL2) guarantees controllability of Λ
- (FL1)-(FL3) are verifiable in terms of f and g_i's using differentiation and algebraic operations only (no need to solve PDE's)
- Geometry: $\mathcal{D}^1 \subset \cdots \subset \mathcal{D}^{n-1} \subset \mathcal{D}^n = TX$.

Assume, for simplicity $m = 1$. Involutivity of $\mathcal{D}^{n-1}$ (of dimension $n-1$ at any x) is equivalent to the existence of a family of hypersurfaces $H_c = \{x \in X : \ h(x) = c\}$ tangent to $\mathcal{D}^{n-1}$



Constructing linearizing transformations

- The normal vector to the hypersurface H_c has to be annihilated by $g, \ldots, ad_f^{n-2}g$ spanning $\mathcal{D}^{n-1}$. So solve

$$(S) \quad \frac{\partial h}{\partial x} A(x) = 0, \text{where } A(x) = (g(x), \ldots, ad_f^{n-2} g(x))$$

- any solution h, $dh \neq 0$ of (S) gives linearizing coordinates

$$z_i = L_f^{i-1} h, \text{ for } 1 \leq i \leq n$$

- and linearizing feedback

$$v = L_f^n h + u L_g L_f^{n-1} h$$

Summary

Mechanical linearization problem

When is the mechanical control system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\Gamma(x,y) + e(x) + g(x)u.\end{aligned} \qquad (\mathcal{MS})$$

where Q is the configuration manifold, $(x,y) \in \mathrm{T}Q$, $u \in \mathbb{R}^m$ is equivalent to a linear mechanical control system on $\mathbb{R}^n \times \mathbb{R}^n$

$$\begin{aligned}\dot{\tilde{x}} &= \tilde{y} \\ \dot{\tilde{y}} &= E\tilde{x} + B\tilde{u}\,?\end{aligned} \qquad (\mathcal{LMS})$$

- $\Gamma(x,y) = \Gamma^i_{jk}(x)y^j y^k$, where Γ^i_{jk} are Christoffel symbols of the affine connection ∇, defining the acceleration $\nabla_{\dot{x}(t)}\dot{x}(t) = (\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x})\frac{\partial}{\partial x^i}$, that correspond to the Coriolis and centrifugal forces
- the vector fields $e(x)$ and $g_r(x)$ correspond to, respectively, uncontrolled and controlled actions on the system.

Why do we need an affine connection ∇? On $Q = \mathbb{R}^2$ the curve

- $\gamma(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ describes a point rotating along the circle $\mathbb{S}^1$
- Its velocity $\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$ is tangent to $\mathbb{S}^1$
- Its acceleration $\begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -\cos(t) \\ -\sin(t) \end{pmatrix}$ points towards the center

In polar coordinates (r, θ)

- $\gamma(t) = \begin{pmatrix} r(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix}$
- Its velocity $\begin{pmatrix} \dot{r}(t) \\ \dot{\theta}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is correct
- But its acceleration $\begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is NOT!

We need an affine connection to define correctly the acceleration as $\nabla_{\dot{x}}\dot{x}$.

Recall feedback linearization

Forget the mechanical structure of $(\mathcal{MS})$ and consider it as

$$\dot{z} = F(z) + G(z)u,$$

whose state $z = (x, y) \in \mathrm{T}Q$. It is feedback linearizable if and only if

(FL1) rk $\mathcal{D}^{2n-1} = 2n$;

(FL2) $\mathcal{D}^j$ is involutive and of constant rank, for $j = 0, 1, \ldots, 2n-1$,

where $\mathcal{D}^j = \text{span}\,\{ad_F^q G_i,\ \ 0 \le q \le j,\ 1 \le i \le m\}$.

Question: Do the linearizing transformations $\tilde{z} = \Phi(z)$ and $u = \alpha(z) + \beta(z)\tilde{u}$ preserve the mechanical structure of $(\mathcal{MS})$

or, equivalently,

Question: are the mechanical and linear structures compatible?

Mechanical feedback transformations

Definition

Consider the group MF of mechanical feedback transformations generated by:
(i) change of coordinates in $\mathrm{T}Q$ *given by* $\Phi : \mathrm{T}Q \to \mathrm{T}\tilde{Q}$ *of the form*

$$(x, y) \mapsto (\tilde{x}, \tilde{y}) = \Phi(x, y) = \left(\phi(x), \frac{\partial \phi}{\partial x}(x) y\right),$$

(ii) mechanical pure feedback transformations, denoted (α, β, γ)*, of the form*

$$u^r = \gamma^r_{jk}(x) y^j y^k + \alpha^r(x) + \beta^r_s(x) \tilde{u}^s,$$

where $\gamma^r_{jk}(x), \alpha^r(x), \beta^r(x)$ *are smooth functions on Q satisfying* $\gamma^r_{jk} = \gamma^r_{kj}$ *and the matrix* (β^r_s) *is of rank m*

Mechanical feedback linearization

Definition

The system $(\mathcal{MS})$ is mechanical feedback linearizable (shortly, MF-linearizable) if there exist a mechanical feedback transformation $(\Phi, \alpha, \beta, \gamma) \in MF$ bringing $(\mathcal{MS})$ into a linear mechanical system of the form $(\mathcal{LMS})$.

For mechanical feedback linearization

- $\tilde{x} = \phi(x)$ maps configurations into configurations;
- the tangent map (Jacobi matrix) maps velocities y into velocities $\tilde{y} = \frac{\partial \phi}{\partial x}(x)y$;
- External forces (controlled or not, potential or not) are mapped into external forces;
- Accelerations $\nabla_{\dot{x}}\dot{x}$ are mapped into accelerations $\tilde{\nabla}_{\dot{\tilde{x}}}\dot{\tilde{x}}$;

In the usual feedback linearization we may not preserve the physical meaning of variables.

The structure of mechanical diffeomorphisms

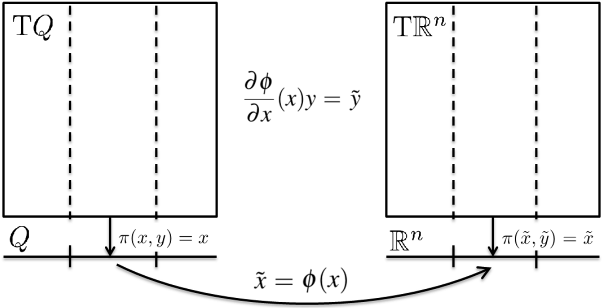


Figure: The structure of a mechanical diffeomorphism Φ.

Trajectories on Q and on $\mathrm{T}Q$

$$\begin{array}{ccc}
z(t, z_0, u) & \xrightarrow{\quad\Phi\quad} & \tilde{z}(t, \tilde{z}_0, u) \\
\Big\downarrow \pi & & \Big\downarrow \pi \\
x(t, z_0, u) & \xrightarrow{\quad\phi\quad} & \tilde{x}(t, \tilde{z}_0, u)
\end{array}$$

Mechanical feedback linearization: tools

- To formulate the result, assume $m = 1$ and associate with $(\mathcal{MS})$ the following sequence of nested distributions

$$\mathcal{E}^0 \subset \mathcal{E}^1 \subset \mathcal{E}^2 \subset \ldots \subset \mathcal{E}^i \subset \ldots \subset \mathrm{T}Q, \text{ where}$$

$$\mathcal{E}^0 = \mathrm{span}\,\{g\}, \quad \mathcal{E}^i = \mathrm{span}\,\{ad_e^j g,\ 0 \leq j \leq i\}$$

are the linearizability distributions of the virtual system $\dot{x} = e(x) + ug(x)$.

Mechanical feedback linearization: conditions

Theorem

Assume $n \geq 3$. A mechanical control system $(\mathcal{MS})$ is, locally around x_0, MF-linearizable to a controllable $(\mathcal{LMS})$ if and only if

$(MF1)$ $\operatorname{rank} \mathcal{E}^{n-1} = n$,

$(MF2)$ $\mathcal{E}^i$ *is involutive and of constant rank, for* $0 \leq i \leq n-2$,

$(MF3)$ $\nabla_{ad_e^i g}\, g \in \mathcal{E}^0 \quad$ *for* $0 \leq i \leq n-1$,

$(MF4)$ $\nabla^2_{ad_e^k g, ad_e^j g}\, e \in \mathcal{E}^1 \quad$ *for* $0 \leq k, j \leq n-1$,

- Conditions $(MF1) - (MF2)$ are the classical feedback linearizability conditions for the system $\dot{x} = e(x) + g(x)u$.
- Conditions $(MF3) - (MF4)$ guarantee that in suitable coordinates linearizing the above system, the vector fields g_r are constant (up to multiplication by β) and the Christoffel symbols Γ^i_{jk} vanish (up to compensation by $\gamma_{jk} y^j y^k$ along the vector field g).

Example: TORA3 system

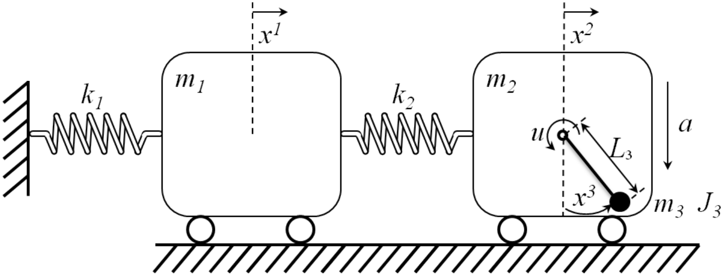


Figure: The TORA3 system

- TORA system (Translational Oscillator with Rotational Actuator) with added gravitational effects is a two dimensional spring-mass system, with masses m_1, m_2 and spring constants k_1, k_2, respectively. A pendulum of length l_3, mass m_3, and moment of inertia J_3 is added to the second body. The control u is a torque applied to the pendulum.
- The kinetic energy is

$$T = \frac{1}{2} m_1 (\dot{x}^1)^2 + \frac{1}{2} (m_2 + m_3)(\dot{x}^2)^2 + \frac{1}{2}(J_3 + m_3 l_3^2)(\dot{x}^3)^2 + m_3 l_3 \cos x^3 \dot{x}^2 \dot{x}^3$$

  so the mass matrix is not constant (the Riemannian metric is not Euclidean but becomes Euclidean in well chosen coordinates).
- Conditions $(MF1) - (MF4)$ are satisfied and the TORA3 system is MF-linearizable.

Dropping the controllability requirement?

- classical results for F-linearization as well as its just presented mechanical counterpart, i.e. MF-linearization, give conditions for transforming a (mechanical) system into a linear controllable (mechanical) system.
- Question: can we generalize them by dropping the requirement of controllability?
- The Riemann tensor R (introduced in the Riemann's habilitation talk, 1854) is defined as the commutator of covariant derivatives

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

It can be written in local coordinates as

$$R = R^i_{jkl} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^l, \text{where}$$

$$R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{pk} \Gamma^p_{jl} - \Gamma^i_{pl} \Gamma^p_{jk}.$$

MF-linearization without controllability assumption

Define 4 annihilators, where $\Lambda(Q)$ is the set of smooth one-forms on Q,

- $\mathrm{ann}\, \mathcal{E}^0 = \{\omega \in \Lambda(Q) : \ \omega(g_r) = 0, \ 1 \le r \le m\}$
- $\mathrm{ann}\, R = \{\omega \in \Lambda(Q) : \ \omega(R) = 0\}$
- $\mathrm{ann}\, \nabla g_r = \{\omega \in \Lambda(Q) : \ \omega(\nabla g_r) = 0, \ 1 \le r \le m\}$
- $\mathrm{ann}\, \nabla^2 e = \left\{\omega \in \Lambda(Q) : \ \omega(\nabla^2 e) = 0\right\}$.

Theorem (Nowicki-R.)

A mechanical system $(\mathcal{MS})$ is, locally around $x_0 \in Q$, MF-linearizable if and only if it satisfies, in a neighborhood of x_0, the following conditions:

$(MFR1)$ $\mathcal{E}^0$ *and* $\mathcal{E}^1$ *are of constant rank,*

$(MFR2)$ $\mathcal{E}^0$ *is involutive,*

$(MFR3)$ $\mathrm{ann}\, \mathcal{E}^0 \subset \mathrm{ann}\, R$,

$(MFR4)$ $\mathrm{ann}\, \mathcal{E}^0 \subset \mathrm{ann}\, \nabla g_r, \quad$ *for* $1 \le r \le m$,

$(MFR5)$ $\mathrm{ann}\, \mathcal{E}^1 \subset \mathrm{ann}\, \nabla^2 e$.

Example: Forced double pendulum on an oscillating base

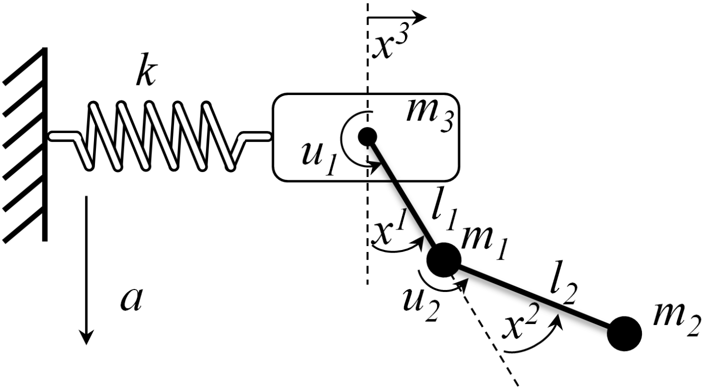


Figure: Forced double pendulum on an oscillating base

Example: Forced double pendulum on an oscillating base

- All conditions (MFR1)-(MFR5) are satisfied so the system is MF-linearizable;
- Although the mass matrix (Riemannian metric) is not Euclidean it becomes Euclidean in well chosen coordinates since $R = 0$ (Riemann theorem).
- (MFR4) and (MFR5) guarantee, respectively, that the control vector fields are constant and the drift $e(x)$ is linear, up to an action of feedback transformations.

Two methods of mechanical linearization

The mechanical drift

$$F = y^i \frac{\partial}{\partial x^i} + (-\Gamma^i_{jk}(x) y^j y^k + e^i(x)) \frac{\partial}{\partial y^i}$$

defines two vector fields: the geodesic spray and $e(x)$ (equivalently, its vertical lift).

- Either linearize the virtual system $\dot{x} = e + ug$ and then make sure that in (suitable) coordinates the Christoffel symbols vanish;
- Or make sure that one can get rid of the Christoffel symbols ($R = 0$) and then make sure that in the Euclidean coordinates the system $\dot{x} = e + ug$ is linear.

Natural questions for future research: mechanical linearization of systems with dissipation

- For nonlinear F-linearizable systems, their normal forms are those of linear systems (Brunovský, controller, controllability etc)
- Are there many F-linearizable nonlinear systems? Unfortunately, NO! Why?
- The single-input system $\Sigma : \dot{x} = f(x) + g(x)u$ is given by $2n$ functions of n variables and the transformation (Φ, α, β) by $n+2$ functions and

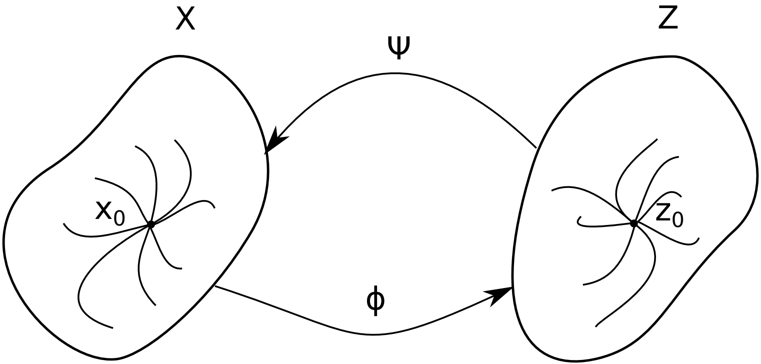
$$n+2 \geq 2n \quad \Leftrightarrow \quad n \leq 2$$

- Thus any control-affine system on $\mathbb{R}^2$ is F-linearizable (out of singularities) and systems on $\mathbb{R}^n$, with $n \geq 3$, are parameterized by $2n - (n-2) = n-2$ functions of n variables.
  Linearizable systems are rare, so precious.
- Two possible directions: either enlarge the class $\mathcal{T}$ of linearizing transformations or us pass to nonlinearizable systems

Summary

Dynamic equivalence

Two systems Ξ and $\tilde{\Xi}$ are dynamically equivalent, shortly D-equivalent, if there exist maps Φ and Ψ mapping trajectories onto trajectories and mutually inverse on trajectories. How to formalize?



Dynamic precompensation

Consider the control system

$$\Xi \ : \ \ \dot{x} = f(x, u), \ \ x \in X \subset \mathbb{R}^n, \ u \in U \subset \mathbb{R}^m$$

together with the precompensation

$$\Pi \ : \ \begin{cases} \dot{y} & = & g(x, y, v), \ \ y \in Y \subset \mathbb{R}^l, \ v \in V \subset \mathbb{R}^m \\ u & = & \psi(x, y, v) \end{cases}$$

The precompensated system becomes

$$\Xi \circ \Pi \ : \ \begin{cases} \dot{x} & = & f(x, \psi(x, y, v)) \\ \dot{y} & = & g(x, y, v). \end{cases}$$

Flatness

Theorem (FLMR, Jakubczyk, Pomet)

The following conditions are equivalent:

(i) *The nonlinear system Ξ is D-equivalent to a linear controllable system Λ;*

(ii) *There exist an endogeneous and invertible precompensator Π for Ξ such that $\Xi \circ \Pi$ is S-equivalent to a linear controllable system Λ;*

(iii) *For $\Xi : \dot{x} = f(x, u), \; x \in X \subset \mathbb{R}^n, \; u \in U \subset \mathbb{R}^m$. there exists m smooth functions $\varphi_i = \varphi_i(x, u, \dot{u}, \ldots, u^{(p)})$, called flat outputs, such that*

$$\begin{array}{rcl} x & = & \gamma(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)}) \\ u & = & \delta(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)}) \end{array}$$

where $\varphi = (\varphi_1, \ldots, \varphi_m)$.

Summary

Definition

A partially linear control system on $\mathbb{R}^n$ is

$$\Lambda_{\mathrm{part}}: \quad \begin{array}{rcl} \dot{z}^1 & = & f^1(z^1, z^2) \\ \dot{z}^2 & = & Az^2 + Bv \end{array}$$

where (z^1, z^2), with $z^1 = (z^1_1, \ldots, z^1_{n-k})$ and $z^2 = (z^2_1, \ldots, z^2_k)$, are coordinates on $\mathbb{R}^n$, $v \in \mathbb{R}$ and the $(k \times k)$-matrix A and the k-vector B are constant.

- A function h is a partially linearizing output of Σ if the relative degree of h is well defined in a neighborhood of x_0, that is, there exists an integer ρ such that $L_g L_f^j h \equiv 0$, for $0 \leq j \leq \rho - 2$, and $L_g L_f^{\rho-1} h(x_0) \neq 0$.
- Define the distributions $\mathcal{D}^j = \mathrm{span}\,\{g, \mathrm{ad}_f g, \ldots, \mathrm{ad}_f^{j-1} g\}$ and let $\overline{\mathcal{D}}^j$ denote the involutive closure of $\mathcal{D}^j$, that is, the smallest involutive distribution containing $\mathcal{D}^j$.

Theorem (Krener-Isidori-Respondek)

The following conditions are equivalent:

- (i) Σ *is locally, around* x_0*, F-equivalent to* Λ_{part}*, with* (A, B) *controllable, where* $\dim z^2 = k$*;*
- (ii) *There exists a part-linearizing output whose relative degree is* k*, around* x_0*.*
- (iii) *There exists a function* h *such that* $\langle \mathrm{d}h, \overline{\mathcal{D}}^{k-1} \rangle = 0$ *and* $\langle \mathrm{d}h, \mathrm{ad}_f^{k-1} g \rangle (x_0) \neq 0$*;*

Moreover, any (and thus all) of the above conditions implies and, under the additional assumption $\mathrm{rank}\, \overline{\mathcal{D}}^{k-1}(x) = const.$*, is equivalent to*

- (iv) Σ *satisfies*
  - (PFL1) $\mathrm{rank}\, \overline{\mathcal{D}}^{k-1}(x) \leq n - 1$ *for any* x *in a neighborhood of* x_0*;*
  - (PFL2) $\mathrm{ad}_f^{k-1} g(x) \notin \overline{\mathcal{D}}^{k-1}(x)$*, for any* x *in a neighborhood of* x_0*;*
  - (PFL3) $g, \ldots, \mathrm{ad}_f^{k-1} g$ *are independent at* x_0*.*

Definition

A system Σ is of class $(n-k, n-r)$ if it is locally F-equivalent to Λ_{part} with k-dim. linear sub-system and not to Λ_{part}, with l-dim. linear sub-system, with $l > k$, and, moreover, $\text{rank}\overline{\mathcal{D}}^{k-1} = const. = r$.

Normal forms

Proposition (Li-Moog-Respondek)

For Σ around $x_0 \in X$, where $\dim X = 4$, we have:

- *If Σ is of class $(1,1)$, then $\mathcal{D}^2$ is not involutive and Σ is locally F-equivalent to*

$$(NF)_{1,1}: \quad \begin{array}{rcl} \dot{z}_1 & = & f_1(z_1) + f_2(z_1, z_2, z_3, z_4) \\ \dot{z}_2 & = & z_3 \\ \dot{z}_3 & = & z_4 \\ \dot{z}_4 & = & v, \end{array}$$

where $f_2(z_1, 0, 0, 0) = 0$ and $\frac{\partial^2 f_2}{\partial z_4^2} \neq 0$.

Proposition (continuation)

For Σ around $x_0 \in X$, where $\dim X = 4$, we have:

- *If Σ is of class $(1,2)$, then $\mathcal{D}^2$ is involutive (but either $\mathcal{D}^3$ is not or the system is not accessible) and Σ is locally F-equivalent to*

$$(NF)_{1,2}: \quad \begin{array}{rcl} \dot{z}_1 & = & f_1(z_1, z_2) + z_3^2 \cdot f_2(z_1, z_2, z_3) \\ \dot{z}_2 & = & z_3 \\ \dot{z}_3 & = & z_4 \\ \dot{z}_4 & = & v, \end{array}$$

  where either $f_2 \not\equiv 0$ or $f_2 \equiv 0$ and $f_1 = f_1(z_1)$.

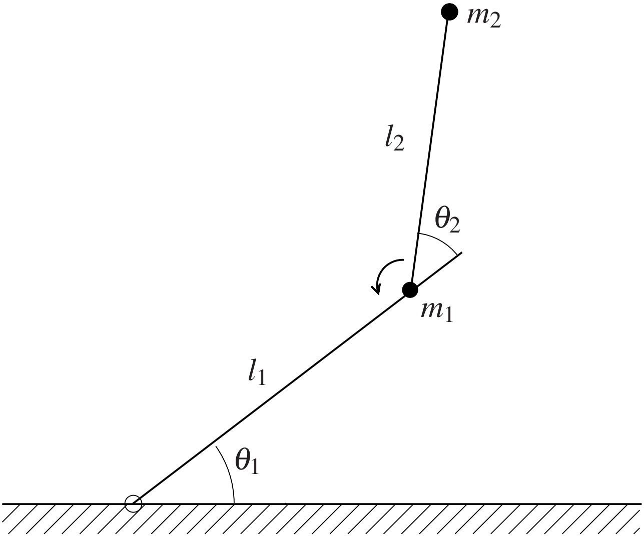


Figure: Acrobot

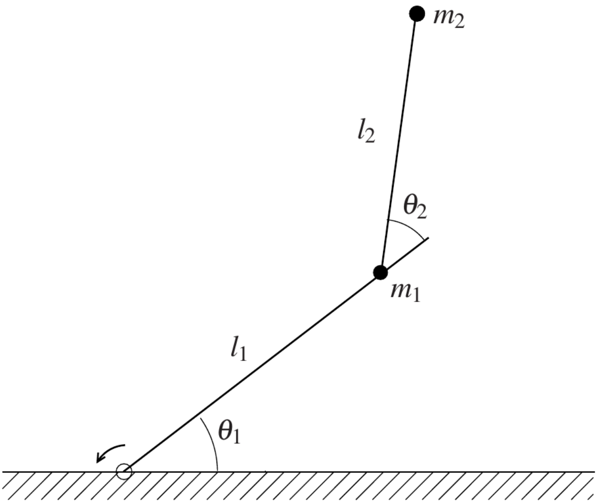


Figure: Pendubot

Corollary

- *Acrobot is a (1,2)-system. In particular, we can find a partially linearizing output with any desired eigenvalue of the corresponding zero dynamics.*
- *Pendubot is a (1,1)-system. In particular, any partially linearizing output leads to the same zero-dynamics.*

Summary

Nonholonomic constraints

- Nonholonomic constraints

$$\Omega(x)\dot{x} = 0,$$

where $x \in \mathbb{R}^n$ and $\Omega = (\Omega^i_j)$, for $1 \leq i \leq k$.

- $x(t)$ satisfies $\Omega(x)\dot{x} = 0$ if an only if

$$\Delta : \dot{x} = \sum_{r=1}^{m} u_r g_r(x), \text{ with } m + k = n,$$

where $< \Omega^i, g_r >= 0$, i.e., $\dot{x} \in \mathcal{G}(x)$ and $\mathcal{G} = \operatorname{span}\{g_1, \dots, g_m\}$.

- Define $\mathcal{G}^0 = \mathcal{G}_0 = \mathcal{G}$ and

$$\mathcal{G}^{j+1} = \mathcal{G}^j + [\mathcal{G}^j, \mathcal{G}^j], \quad b^j = \operatorname{rk} \mathcal{G}^j$$
$$\mathcal{G}_{j+1} = \mathcal{G}_j + [\mathcal{G}_0, \mathcal{G}_j], \quad s_j = \operatorname{rk} \mathcal{G}_j.$$

Chained form

Theorem

The following conditions are equivalent:

(i) Δ *is F-equivalent to the chained form*

$$\begin{aligned} \dot{z}_1 &= v_1 & \dot{z}_2 &= z_3 v_1 \\ & & \dot{z}_3 &= z_4 v_1 \\ & & &\vdots \\ & & \dot{z}_{n-1} &= z_n v_1 \\ & & \dot{z}_n &= v_2 \end{aligned}$$

(ii) Δ *is dynamically linearizable;*

(iii) Δ *is flat;*

(iv) $b^j = s_j$, *for* $0 \le j \le n-2$, *and* $(b^0, b^1, \ldots, b^{n-2}) = (2, 3, \ldots, n)$.

Chained form

When can we bring Δ to the chained form?

$\Delta: \ \dot{x} = u_1 g_1(x) + u_2 g_2(x)$ is given by $2n$ functions (of n variables), while (Φ, β) is given by $n + 2 \cdot 2 = n + 4$ functions, so

$$n + 4 \geq 2n \Leftrightarrow n \leq 4.$$

Thus out of singularities, Δ on $\mathbb{R}^3$ and $\mathbb{R}^4$ are F-equivalent to the chained form but on $\mathbb{R}^5$, $n \geq 5$, this requires $n - 4$ functional conditions.

Bringing to a normal form

- Take a vector $v \in \mathbb{R}^n$. If $v \neq 0$, then there exists an invertible matrix $A \in M^{n \times n}$ such that

$$Av = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

- Take a vector field $v(x)$ on $\mathbb{R}^n$. If $v(x) \neq 0$, then (locally) there exists $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\frac{\partial \Phi}{\partial x}(x) v(x) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

- Take a co-vector $\omega \in (\mathbb{R}^n)^*$. If $\omega \neq 0$, then there exists an invertible matrix $A \in M^{n\times n}$ such that

$$\omega A = \begin{bmatrix} 1, & 0, & \cdots & ,0 \end{bmatrix}$$

- Take a differential 1-form $\omega(x)$ on $\mathbb{R}^n$. If $\omega(x) \neq 0$, then (locally) there exists $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\omega(x)\frac{\partial \Phi}{\partial x}(x) = \begin{bmatrix} 1, & 0, & \cdots & ,0 \end{bmatrix}$$

  if and only if ω is closed=exact, i.e., ω is the differential of a function.

- Point-wise duality between vectors and co-vectors is not maintained locally: the nature of vector fields and differential 1-forms are different.

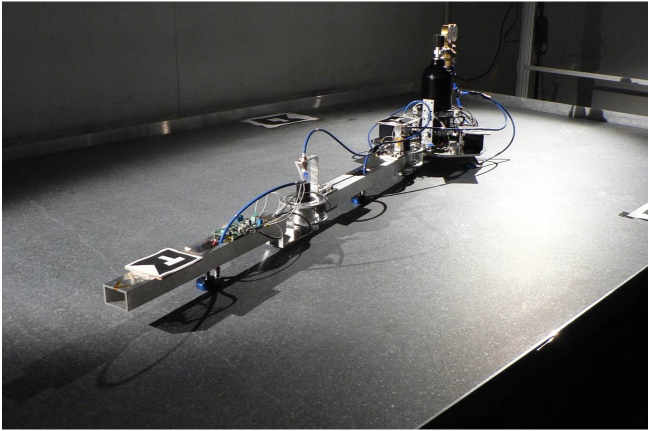


Figure: SRC space manipulator

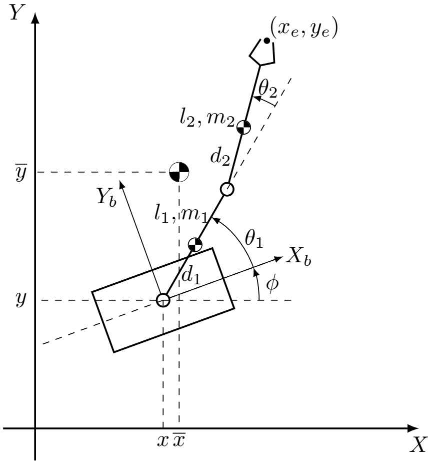


Figure: SRC space manipulator

SRC space manipulator

- model of a space manipulator built recently in the Space Research Centre (SRC) of the Polish Academy of Sciences
- composed of a mobile base (a satellite) and a 2DOF planar on-board manipulator. To simulate on Earth the lack of gravity, the manipulator is supported by air bearings that enable it to float horizontally over a granite table
- generalized coordinates are $\bar{q} = (\bar{x}, \bar{y}, \phi, \theta_1, \theta_2)^T \in \mathbb{R}^2 \times \mathbb{T}^3$, where bars refer to barycentric position coordinates (the position of the centre of mass) and $\mathbb{T}^3 = S^1 \times S^1 \times S^1$ denotes the 3-dimensional torus
- the linear momenta and the angular momentum are preserved and we denote their constant values by, respectively, p_1, p_2, and p_3.
- The center of mass of the manipulator moves uniformly along a straight line,

$$\mathrm{A}\dot{\bar{x}} = \mathrm{p}_1, \quad \mathrm{A}\dot{\bar{y}} = \mathrm{p}_2.$$

- The conservation of the angular momentum results in

$$F(\theta_2)\dot{\phi} + G(\theta_2)\dot{\theta}_1 + H(\theta_2)\dot{\theta}_2 = \mathrm{p}_3.$$

- Denote $q = (\theta_1, \theta_2, \phi)^T \in \mathbb{T}^3$. Then, the conservation law either gives the affine nonholonomic constraint

$$\mathcal{A}(q)\dot{q} = \mathrm{p}_3, \quad \text{when } \mathrm{p}_3 \neq 0,$$

or the nonholonomic constraint

$$\mathcal{A}(q)\dot{q} = 0, \quad \text{if } \mathrm{p}_3 = 0.$$

- Affine nonholonomic constraint can be represented by the control-affine system

$$\dot{q} = f(q) + u_1 g_1(q) + u_2 g_2(q).$$

and linear nonholonomic constraint by the driftless system

$$\dot{q} = u_1 g_1(q) + u_2 g_2(q),$$

where $\mathcal{A}(q)g_i(q) = 0$ and $f(q) = \mathcal{A}^{\#}(q)b(q)$, with $\mathcal{A}^{\#}(q)$ denoting a right inverse of $\mathcal{A}(q)$

Normal forms for the SRC space manipulator, case $p_3 \neq 0$

Theorem

Consider the SRC manipulator Σ.

1. *Suppose that* $p_3 \neq 0$. *Around any* $q \in \mathbb{T}^3$, *such that* $\theta_2 \neq 0, \pi$ *(i.e., outside control distribution singularities S) the control-affine system* Σ *is locally feedback equivalent to the normal form*

$$\dot{z}_1 = v_1,\ \dot{z}_2 = v_2,\ \dot{z}_3 = 1 + z_1 v_2. \tag{1}$$

2. *Suppose that* $p_3 \neq 0$. *Around any* $q \in \mathbb{T}^3$, *such that* $\theta_2 = 0, \pi$ *(i.e., in a neighbourhood of a control distribution singularity S) the control-affine system* Σ *is locally feedback equivalent to the normal form*

$$\dot{z}_1 = v_1,\ \dot{z}_2 = v_2,\ \dot{z}_3 = 1 + z_1^2 v_2. \tag{2}$$

Normal forms for the SRC space manipulator, case $p_3 = 0$

Theorem

1. *Suppose that* $p_3 = 0$. *Around any* $q \in \mathbb{T}^3$, *such that* $\theta_2 \neq 0, \pi$ *(i.e., outside control distribution singularities (S)) the driftless system* Σ *is locally feedback equivalent to the normal form*

$$\dot{z}_1 = v_1, \ \dot{z}_2 = v_2, \ \dot{z}_3 = z_1 v_2. \tag{3}$$

2. *Suppose that* $p_3 = 0$. *Around any* $q \in \mathbb{T}^3$, *such that* $\theta_2 = 0, \pi$ *(i.e., in a neighbourhood of a control distribution singularity S) the driftless system* Σ *is locally feedback equivalent to the normal form*

$$\dot{z}_1 = v_1, \ \dot{z}_2 = v_2, \ \dot{z}_3 = z_1^2 v_2. \tag{4}$$

- The geometry of the normal forms is determined by

$$S(q) = \{q \in Q : \det(g_1, g_2, [g_1, g_2])(q) = 0\},$$

where $[g_1, g_2]$ is the Lie bracket of the vector fields g_1 and g_2.
- The set $S = \{\theta_2 = 0\} \cup \{\theta_2 = \pi\}$ consists of singularities of the control distribution $\mathcal{G}$ and is formed by the union of two tori.
- For $\mathrm{p}_3 \neq 0$ (resp. $\mathrm{p}_3 = 0$), we get two normal forms (1) and (2) (resp. 3 and (4)) depending on whether $q \notin S$ or $q \in S$.

- The normal form

$$\dot{z}_1 = v_1 \qquad \begin{aligned} \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_1 v_2. \end{aligned}$$

  is the simplest case of the chained form
- Any driftless $(n, m) = (3, 2)$-system is (locally) feedback equivalent to the chained form, out of singularities
- Only one local normal form for driftless $(n, m) = (3, 2)$-systems: unicycle, skate, knife edge, space robot with zero angular momentum are locally feedback equivalent to the chained form
- Only one normal form (chained form) for driftless $(n, m) = (4, 2)$-systems (nonholonomic car, rolling penny)
- For $n \geq 5$, many nonequivalent driftless systems (functional paramaters). The chained form becomes very rare (of codimension infinity)

Summary

Crane system

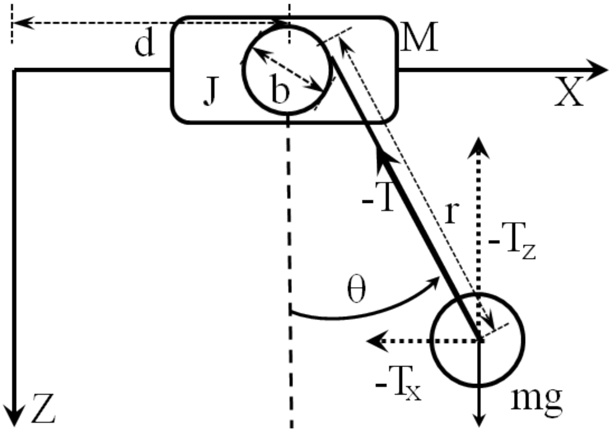


Figure: Crane

Crane system

The dynamics of the system are

$$\begin{aligned} \mu\ddot{x} &= -T\sin\theta \\ \mu\ddot{z} &= -T\cos\theta + \mu g \\ M\ddot{d} &= -c_d\dot{d} + \tilde{\mathcal{F}} + T\sin\theta = \mathcal{F} + T\sin\theta \\ J\ddot{r} &= -c_r\dot{r} - b\tilde{\mathcal{C}} + b^2T = \mathcal{C} + b^2T,, \end{aligned} \tag{5}$$

that are subject to the following constraints:

$$\begin{aligned} x &= r\sin\theta + d \\ z &= r\cos\theta. \end{aligned} \tag{6}$$

- T is the tension of the rope
- $\tilde{\mathcal{F}}$ is an external force that is controlled
- $b\tilde{\mathcal{C}}$ is the external momentum, that serves as a second control in the system

Eliminating the tension T

- The system evolves on 8-dimensional state-space (4 configurations and 4 velocities, θ can be eliminated via (6)).
- and three driving variables (free variables, i.e. differentially unconstrained variables) are $(\mathcal{F}, \mathcal{C}, T)$, and subject to one holonomic constraint acquired from (6):

$$h := (x-d)^2 + z^2 - r^2 = 0, \tag{7}$$

  which describes a cone in $\mathbb{R}^2 \times \mathbb{R}_+$, translated by d along the variable x, the apex being excluded by $r > 0$.
- We can consider the tension T as a Lagrange multiplier and eliminate it to get the evolution on a 6-dimensional state-space.
- Equivalently we can calculate the zero dynamics corresponding to $h =$ with respect T considered as a variable controlled by "the Nature".

Normal form for the 2-crane

Proposition (Kozłowski, Nowicki, Piasek, R.)

The 2-crane system $TQ^{\pm}$, where $Q^{\pm} = \{(x, z, d) \in \mathbb{R}^3 \ z \neq 0$ is globally static feedback equivalent on Q^{+} and on Q^{-} to the form

$$\begin{aligned} \ddot{x} &= u_2(x - d) \\ \ddot{z} &= u_2 z + g \\ \ddot{d} &= u_1 - u_2 \frac{\mu}{M}(x - d). \end{aligned}$$

and to the second order chained form with a constant drift vector field:

$$\begin{aligned} \ddot{z}_1 &= v_1 \\ \ddot{z}_2 &= v_2 + g \\ \ddot{z}_3 &= z_1 v_2. \end{aligned}$$

Normal form for the *m*-crane

Proposition (Kozłowski, Nowicki, Piasek, R.)

The m-crane system on $TQ^{\pm}$, *where* $Q^{\pm} = \{(x_i, x_m, d_i) \in \mathbb{R}^{2m-1} : x_m \neq 0,\ 1 \leq i \leq m-1\} = Q^{+} \cup Q^{-}$ *is globally static feedback equivalent on* Q^{+} *and on* Q^{-} *to*

$$\begin{aligned} \ddot{x}_i &= u_m(x_i - d_i), && \text{for } 1 \leq i \leq m-1, \\ \ddot{x}_m &= u_m x_m + g \\ \ddot{d}_i &= u_i - u_m \frac{\mu_m}{\mu_i}(x_i - d_i), && \text{for } 1 \leq i \leq m-1, \end{aligned} \tag{8}$$

and to the second order chained form with a constant drift vector field:

$$\begin{aligned} \ddot{z}_i &= v_i && \text{for } 1 \leq i \leq m-1 \\ \ddot{z}_m &= v_m + g \\ \ddot{z}_{m+i} &= z_i v_m && \text{for } 1 \leq i \leq m-1. \end{aligned} \tag{9}$$

Flatness of the *m*-crane systems

Proposition (Kozłowski, Nowicki, Piasek, R.)

1. *The m-crane system is globally config-flat on* $TQ^{\pm}$, *provided that the control* $u \in \mathbb{R}^m$ *satisfies* $u_m \neq 0$, *with a global flat output* $\phi = (\phi_1, \ldots, \phi_m) = (x_1, \ldots, x_m)$.
2. *The two-dimensional dynamic precomensator* $\ddot{u}_m = v_m$, *together with a suitable static feedback transformation (applied to the precompansated system) transform the m-crane system into the linear controllable system (normal form under dynamic feedback)*

$$\ddot{x}_i = v_i, \quad \mathit{for} \quad 1 \leq i \leq m.$$

- The trajectory tracking problem for any *m*-crane system is thus reduced to that for the above linear system (transformations bringing the *m*-crane system into its normal form are explicitly computable).

Summary

Conclusions

- We presented normal forms for several classes of systems
- Normal forms allow, from one side, to understand the nature and geometry of the systems and, on the other hand, are very useful in applications
- Do not confuse linearization (static, dynamic) with linear approximation
- Whenever we can linearize the system (statically, dynamically) or bring it to a normal form, the control problems, we are dealing with, get substantially simplified
- Even if we do not apply linearizing transformations or the system is not linearizable (flat), our knowledge about the system is deeper: we identify intrinsic nonlinearities that cannot be removed via feedback (static, dynamic)
- linearization (more generally, normal forms) under non-invertible feedback is difficult since geometric invariants are not preserved.

Conclusions: what do we know about flatness?

- Via flatness (dynamic linearization) we can solve the constructive controllability problem
- Although very useful, flatness is a highly non generic property: a slight perturbation of a flat system yields a non flat one (Tchoń)
- We know that a few classes of control systems are flat: accessible systems with $n-1$ controls, accessible control-linear systems with $n-1$ and $n-2$ controls
- We know to characterize flat control systems of special forms: feedback linearizable systems, control-linear systems with 2 controls (chained form), m-chained form
- or of very special dimensions: 3 states and 2 controls (nonlinear) and 4 states and 2 controls (affine)
- We know to characterize flat systems of differential weight $n+m+1$

Conclusions: what don't we know about flatness?

- We do not know to characterize flatness in general.
- We do not know whether the problem is finite or infinite dimensional, that is, we do not know if there is a bound on the number of derivatives of controls
- We do not even know how to check flatness for control-affine systems with 2 controls nor for control-linear systems with 3 controls
- We know that the problem is difficult: Ellie Cartan (1914) has introduced the notion of absolute equivalence of underdetermined differential equations. His absolutely trivial equations are just flat systems. He proved that systems with 2 controls are flat (absolutely trivial) if and only if they are equivalent to the chained form (Goursat normal form). Cartan claimed that the general problem is difficult.
- Non flat systems exist! The first example is due David Hilbert (1912) who had also been working on absolute equivalence (integrating differential equations without integration). His example is, geometrically, the same as the unicycle towing a trailer but with a hook that is not at the mid-point.